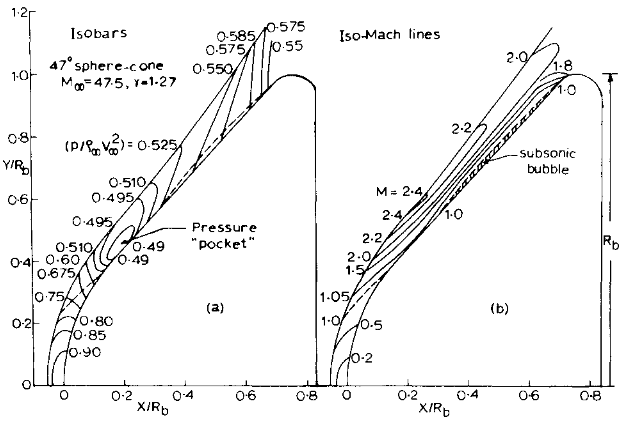

Fig. 3 Isobars and iso-Mach lines for a 47° half-angle sphere-cone.

adverse pressure gradient monotonically decreases with cone angle, its magnitude will attain a maximum value for cone angles in the vicinity of 47°.

Figure 2 also presents the variation of shock angle with radial distance from the axis. The shock shape up to the sonic point is unaffected until the cone angle is close to the detachment angle of a sharp cone. The location of the shock sonic point rapidly changes with further increase in the cone angle. The concavity of the shock in the sphere-cone junction region reveals the possibility of having three shock sonic points for certain cone angles between 60° and 70° to include a small region of supersonic flow in the vicinity of the inflection point.

Interesting observations will be made from the contours of isobars and iso-Mach lines in the shock layer of a 47° sphere-cone. Figure 3 depicts the presence of a pressure "pocket" near the surface, which is attributed to the proximity of the inflection point to the sphere-cone junction. This pressure pocket disappears at both small and large cone angles. The iso-Mach lines in the spherical portion of the body exhibit a monotonic decrease of Mach number from the shock to body surface. However, the Mach number in the conical portion increases to a maximum value and then decreases as the body surface is approached.

The present results demonstrate the capability of the time-asymptotic method, with proper accounting for the entropy layer, to treat a wide range of sphere-cone bodies. In addition, the supersonic flow on the conical part of a small angle body changes into subsonic flow at large angles with the appearance and spreading of a subsonic bubble on the surface at intermediate cone angles. This pronounced change in the flow characteristics significantly modifies the shock wave and sonic line shapes, and results in the appearance of a pressure pocket and an adverse pressure gradient at the junction region.

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Large Amplitude Forced Vibrations of Elastic Structures

L. W. Rehfield*

Georgia Institute of Technology, Atlanta, Ga.

Introduction

Aerospace vehicle structures are largely composed of thin-walled elements stiffened by beam-like members or made of composite construction. These elements are often caused to vibrate at large amplitudes during certain phases of flight. Consequently, large amplitude vibrations are of considerable interest and a number of papers have been written on the subject. An extensive bibliography of recent papers published on large amplitude flexural vibrations of thin elastic plates and shells has been compiled by Pandalai.[1]

A general approach to nonlinear free vibrations of elastic structures has been developed by the present author.[2,3] The approach is based upon Hamilton's principle and a perturbation procedure and is in much the same spirit as Koiter's theory of initial postbuckling behavior.[4,5] It provides information regarding the first-order effects of finite displacements upon the frequency and dynamic stresses arising in free, undamped vibration of structures.

In the present Note the general approach just described is extended to the technically important case of forced vibrations. The structure is assumed to be excited by a disturbing force which is a harmonic function of time and is spatially distributed so as to drive only a single mode. The theory reduces to that of free vibrations in the absence of a disturbing force. Also, linearized vibration theory is recovered for small amplitude motion. Applications to the forced, undamped vibrations of beams and rectangular plates are presented as illustrations.

Basic Equations

The functional notation of Budiansky[6] will be utilized to permit a concise presentation of the theory. The applied loading q produces generalized displacement u, strain γ, and stress σ. The dynamics of the system is established by Hamilton's principle, which, for periodic motion with circular frequency ω, can be symbolically written[7]

$$\int_0^{2\pi} [\omega^2 M(\ddot{u})\cdot\delta u+\sigma\cdot\delta\gamma-q\cdot\delta u]\,d\tau = 0 \tag{1}$$

The "dot" operation signifies the appropriate inner multiplication of variables and integration of the result over the entire structure. The generalized mass operator M is assumed to be homogeneous and linear with the property that

$$M(u)\cdot v = M(v)\cdot u \tag{2}$$

for all u and v. $\tau = \omega t$ is the time variable, t is time and $(\dot{\ }) = \partial(\ )/\partial\tau$. δu is any virtual displacement that is consistent with the kinematic boundary conditions imposed on the structure.

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* Associate Professor of Aerospace Engineering. Member AIAA.

Equation (1) is supplemented by the field equations

$$\gamma = e + \tfrac{1}{2}L_2(u) \tag{3}$$

$$\sigma = H(\gamma) \tag{4}$$

e is the linearized strain measure and L_2 is a homogeneous quadratic functional. For Hookean elastic structures, H is a homogeneous linear functional. The following reciprocity relation

$$\sigma^{(1)} \cdot \gamma^{(2)} = \sigma^{(2)} \cdot \gamma^{(1)} \tag{5}$$

is assumed also; "1" and "2" are any arbitrary states of stress and strain. In addition, the homogeneous bilinear functional L_{11} is defined by the following equation:

$$L_2(u+v) = L_2(u) + 2L_{11}(u, v) + L_2(v) \tag{6}$$

It follows that $L_{11}(u, v) = L_{11}(v, u)$ and $L_{11}(u, u) = L_2(u)$. If use is made of the preceding definition, the variation of the generalized strain can be written as

$$\delta\gamma = \delta e + L_{11}(u, \delta u) \tag{7}$$

Synopsis of Free Vibration Theory

The equations governing free, undamped vibrations are obtained by setting $q = 0$ in Eq. (1). The vibration modes and frequencies of linearized theory can be found by setting

$$u = \xi u_1, \quad \gamma = \xi e_1, \quad \sigma = \xi\sigma_1 \tag{8}$$

and retaining only linear terms in ξ. ξ is an amplitude parameter associated with the mode u_1 which has natural frequency ω_o. The result is

$$\int_0^{2\pi} [\omega_o^2 M(\ddot{u}_1) \cdot \delta u + \sigma_1 \cdot \delta e]\, d\tau = 0 \tag{9}$$

Equating the integrand to zero yields the linearized equation of motion.

We assume at this point that a single mode u_1 is associated with the natural frequency ω_o. The case of multiple modes corresponding to the same natural frequency will not be treated here.

To discover how the structure behaves for finite amplitudes, we assume

$$\begin{aligned} u &= \xi u_1 + \xi^2 u_2 + \cdots \\ \gamma &= \xi e_1 + \xi^2[e_2 + \tfrac{1}{2}L_2(u_1)] + \cdots \\ \sigma &= \xi\sigma_1 + \xi^2\sigma_2 + \cdots \end{aligned} \tag{10}$$

In order to make the expansions unique, additional conditions must be imposed. It is convenient to orthogonalize the displacement increments $u_2, u_3, \ldots$ with respect to u_1 in the sense that†

$$\int_0^{2\pi} M(\dot{u}_1) \cdot \dot{u}_k\, d\tau = \int_0^{2\pi} M(\ddot{u}_1) \cdot u_k\, d\tau = 0 \qquad (k \neq 1) \tag{11}$$

The substitution of Eqs. (10) into Eq. (1) (with $q = 0$), the collection of terms in powers of ξ, and the substitutions $\delta u = u_1$, $\delta e = e_1$ ultimately result in

$$(\omega^2/\omega_o^2) = 1 + A\xi + B\xi^2 + \cdots \tag{12}$$

where

$$A = \frac{\int_0^{2\pi} \tfrac{3}{2}\sigma_1 \cdot L_{11}(u_1, u_1)\, d\tau}{\omega_o^2 \int_0^{2\pi} M(\dot{u}_1) \cdot \dot{u}_1\, d\tau} \tag{13}$$

and

$$B = \frac{\int_0^{2\pi} [2\sigma_1 \cdot L_{11}(u_1, u_2) + \sigma_2 \cdot L_{11}(u_1, u_1)]\, d\tau}{\omega_o^2 \int_0^{2\pi} M(\dot{u}_1) \cdot \dot{u}_1\, d\tau} \tag{14}$$

Equation (12) provides an asymptotic relation for the frequency in terms of the amplitude of the linear vibration mode; the details of its derivation may be found in Refs. 2, 3, or 7.

In the evaluation of B a solution for u_2 and σ_2 is required. The governing variational equation is[2,3,7]

$$\omega_o^2 M(\ddot{u}_2) \cdot \delta\tilde{u} + \sigma_2 \cdot \delta\tilde{e} + \sigma_1 \cdot L_{11}(u_1, \delta\tilde{u}) = 0 \tag{15}$$

where $\delta\tilde{u}$ is orthogonal to u_1 in the sense of Eq. (11). Also

$$\gamma_2 = e_2 + \tfrac{1}{2}L_{11}(u_1, u_1) \tag{16}$$

and

$$\sigma_2 = H(\gamma_2) \tag{17}$$

Forced Vibration Theory

The transition from free to forced vibrations is not smooth and continuous. While free vibrations lead to bifurcation phenomena, forced vibrations do not.‡ This situation is similar to the one encountered in postbuckling theory when geometric imperfections are introduced. We will obtain a general expression for a single driven vibration mode in much the same way that Budiansky[6] treats geometric imperfections in Koiter's postbuckling theory.

We restrict our attention to disturbing forces of the form

$$q = \lambda q_o(\mathbf{r}, t) \tag{18}$$

where λ is a loading magnitude parameter, $\mathbf{r}$ is the position vector, and q_o is the distribution of the loading. q_o is taken to be a harmonic function of time with circular frequency ω and is spatially distributed so as to drive only the mode u_1. ω is in the neighborhood of ω_o, the natural circular frequency of the mode u_1. The governing equation is

$$\int_0^{2\pi} [\omega^2 M(\ddot{u}) \cdot \delta u + \sigma \cdot \delta\gamma - \lambda q_o \cdot \delta u]\, d\tau = 0 \tag{19}$$

Any solution to Eq. (19) must satisfy two physical conditions. First, it must reduce to the free vibration results for $\lambda = 0$, and, second, linear vibration theory should be recovered for small amplitudes. An approximate solution which satisfies these two conditions can be obtained by a Galerkin-type process using the solution (10) for free vibrations. We set $\delta u = u_1$ and assume the solution (10); this results in

$$\xi[1 - (\omega^2/\omega_o^2)] + \xi^2 A + \xi^3 B + \cdots = \lambda\phi_o \tag{20}$$

where

$$\phi_o = \int_0^{2\pi} q_o \cdot u_1\, d\tau \Big/ \omega_o^2 \int_0^{2\pi} M(\dot{u}_1) \cdot u_1\, d\tau \tag{21}$$

Equation (20) extends our theory to forced vibrations in a single mode. It will subsequently be applied to representative examples of beam and plate structures.

Applications

Consider a uniform, Hookean elastic beam of symmetric cross section with simply supported immovable ends. An axial force is induced in the beam due to finite amplitude vibrations because the supports are assumed to be immovable. We have studied the forced vibration of this structure for a class of lateral distributing forces of the form

$$p(X, t) = Q \sin(k\pi X/L) \cos\omega t \tag{22}$$

where p is the distributed loading per unit length, L is the beam length, k is an integer, Q is the magnitude of the loading, and X is the longitudinal coordinate along the beam span.

Large amplitude free vibrations[2] in the kth natural vibration mode are governed by

$$\frac{\omega^2}{\omega_{ok}^2} \approx 1 + \frac{3\pi}{16}\xi^2 \tag{23}$$

where

$$\omega_{ok}^2 = (k^4\pi^4 EI/mL^4) \tag{24}$$

E is Young's modulus, I is the second moment of area, and m

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† These orthogonality conditions differ from those of Ref. 2 in that they involve integration with respect to τ.

‡ An exception is the appearance of subharmonics or superharmonics in certain instances.[8]

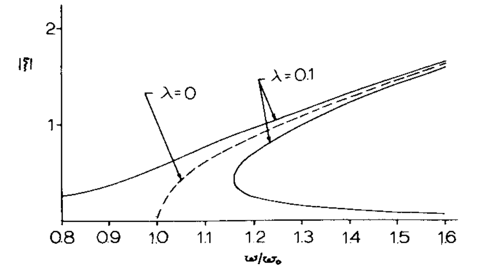

Fig. 1 Nonlinear forced response of a simply supported beam for $k = 1$, the fundamental mode.

is the mass per unit length of the beam. ξ is a dimensionless amplitude parameter defined as

$$\xi = A/(\pi\rho)^{1/2} \tag{25}$$

where A is the maximum amplitude of the lateral deflection and ρ is the radius of gyration of the beam cross section. Since the natural vibration modes are sinusoidal functions of X of the same form as p, it is a straightforward matter to determine the forced response; the result is[7]

$$\xi\left(1-\frac{\omega^2}{\omega_{ok}{}^2}\right)+\frac{3\pi}{16}\xi^3 \approx \frac{\lambda}{k^4} \tag{26}$$

λ is a dimensionless loading parameter defined as $Q/(\pi m\rho\omega_{o1}{}^2)^{1/2}$.

A frequency response curve based upon Eq. (26) for $k = 1$, the fundamental mode, is shown in Fig. 1. The backbone curve for free vibration is also plotted in the figure. The results are typical for a hardening system without damping. Identical results have also been obtained[7] by the method of averaging[9] and by Duffing's method[10]; this strengthens our confidence in the theory as a useful approach for forced vibrations in general.

A related problem is that of the lateral forced vibration of a simply supported rectangular plate with immovable edges. The plate is uniform and Hookean elastic with length a, width b, and thickness h. If the edges are immovable, membrane stresses will be induced in the plate due to transverse flexural vibrations of finite amplitude. For forced vibration in the fundamental mode, we consider a disturbing distributed loading of the form

$$p(X, Y, t) = Q\sin(\pi X/a)\sin(\pi Y/b)\cos\omega t \tag{27}$$

where here p and Q have the units of force per unit area. X, Y are coordinates.

Nonlinear free vibrations are considered in Ref. 2. Forced vibrations in the fundamental mode due to the distributed loading (27) turn out to be governed by[7]

$$\xi\left(1-\frac{\omega^2}{\omega_o{}^2}\right)+\xi^3\frac{3}{32(\mu^2+1)^2}\left[\frac{(\mu^4+2\nu\mu^2+1)}{(1-\nu^2)}+\frac{1}{2}(\mu^4+1)\right] \approx \frac{\lambda}{(\mu^2+1)^2} \tag{28}$$

where

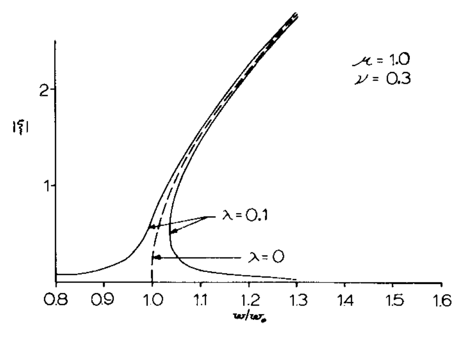

Fig. 2 Nonlinear forced response of a simply supported square plate in its fundamental mode.

$$\xi = [12(1-\nu^2)]^{1/2}\frac{A}{h} \qquad \omega_o{}^2 = \frac{(\mu^2+1)^2}{12(1-\nu^2)}\frac{Eh^2}{\rho}\left(\frac{\pi}{b}\right)^4$$
$$\lambda = \{[12(1-\nu^2)]^{3/2}/E\}(b/\pi h)^4 Q \tag{29}$$

$\mu = b/a$ is the plate aspect ratio, ν is Poisson's ratio, ρ is the mass density, and A is the maximum amplitude of the lateral deflection.

A frequency response curve based upon Eq. (28) for $\mu = 1$, the square plate, and $\nu = 0.3$ is shown in Fig. 2. The backbone curve for $\lambda = 0$ is also shown in the figure. Again, the results are typical for a hardening system without damping.

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Response of a Three-Layered Ring to an Axisymmetric Impulse

MATHIAS J. SAGARTZ*
Sandia Laboratories, Albuquerque, N.Mex.

THE axisymmetric transient response of a three-layered shell having a thick, soft middle layer has received the attention of several investigators. In Ref. 1 solutions were developed for a circular elastic shell supported by an elastic core. This analysis considered propagation and reflection of stress waves in the middle layer or core; and the inner core boundary was specified as a rigid reflector, a free surface, or a support offered by another shell. These solutions were limited to three wave travel times through the elastic core because the algebraic computations became more complicated with each additional wave transit time through the core. This Note presents a modal solution which is not restricted to three wave transit times and which is suitable for studying the stiffening effect of the core and inner shell on the outer shell.

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* Staff Member, Shock Simulation Department.